Generalized Compress-and-Forward Strategy For Relay Networks

Mohammad Hossein Yassaee, Mohammad Reza Aref
Information Systems and Security Lab (ISSL)
EE Department, Sharif University of Technology, Tehran, Iran
E-mail: yassae@ee.sharif.edu, aref@sharif.edu

Abstract—In this paper, we present a new generalization of the well-known Compress-and-Forward strategy for relay networks. We propose an offset decoding at destination, where destination considers an ordered partition of relays and decodes information of any partition with the help of information from prior partitions. We show that when we do not partition the set of relays, our result improves the result of Kramer, et al. A geometrical method is utilized to unify results of different ordered partitioning. Also, the unified result has a celebrated source-channel coding separation interpretation.

I. INTRODUCTION

In the last decade, relay networks have attracted more attention (see [2] and references therein). In a relay network [5], a source communicates with a destination with the help of arbitrary number of relays. For this network, two basic strategies are proposed by Cover and El Gamal [1]. First strategy, called decode-and-forward (DF) has been studied widely in [3] and references therein. The second strategy was called compress-and-forward (CF) where the relays only transmit the compressed version of their channel outputs to destination. CF was studied and generalized by Kramer et al in [2]. In [6] a time sharing assignment applied to CF strategy and a computational achievable rate was derived. Also a mixed strategy for relay networks when the relays either use DF or CF was proposed in [2]. Recently, Ghabeli and Aref generalized relay networks when the relays do not use DF or CF was proposed in [2]. Recently, Ghabeli and Aref generalized partial decode-and-forward (PDF) to relay networks using regular encoding/backward decoding [7]. In [8], Rost and Fettweis generalize the mixed strategy of Cover and El Gamal based on the generalization of the PDF. In addition, their scheme uses the ideas of successive refinement with different side information atreceiver.

In [4], authors proposed an offset encoding technique for Multiple Access Relay channel (MARC) such that it recovers the corner points of destination’s backward decoding rate regions. Therefore, every point on the boundary of backward decoding rate regions is achievable by a timesharing between corner points. Also, they showed that at least for the two-source case, time sharing is not needed if both offset encoding and non-offset encoding are used. Xie and Kumar [3] showed that when we have more than one constraint on the rate region, except for some simple scenario, timesharing can not recover all points of backward decoding rate region. So, as an open question, Xie and Kumar [3] and [4] asked when backward decoding can be replaced with a decoding strategy with less delay (such as sliding window decoding or successive decoding; etc). In this paper, we partially treat this problem.

In this paper, we focus on generalizing the compress and forward strategy. In [2], the destination first decodes the bin index of all compressed version of channel outputs through a Multiple Access Channel (MAC), then determines the compressed signal. However in a general relay network, all relay-destination channels do not have similar strength. For example, perhaps there is a relay close to the destination such that the compressed signal of the relay can be transmitted through a direct channel between it and the destination without help of the other relays. In other words, assume there exists a relay that satisfies the feasibility constraint of Cover and El Gamal’s CF strategy for single relay channel. Now by decoding the information of this relay, there are additional information at the destination. So, destination can decode the information of other relays with more reliability. This view motivates us to define an ordered partition on the set of relays, such that each partition consists of relays with similar situation. For each ordered partition, we define an offset decoding technique at destination, then we compute its feasibility region. We show that when we do not partition, our computation leads to a more flexible constraint than [2] for the CF strategy. The result of offset decoding is complicated and gives limited insight. A novel geometrical method is utilized to unify these regions. Also, a separation of source coding and channel coding interpretation is given for our result.

In the unifying feasibility regions, we see that our proposed problem can be regarded as a special case of the general problem of replacing backward decoding rate region with a decoding scheme with less delay proposed by Xie and Kumar in [3].

II. NOTATION AND DEFINITION

A. Notation

We denote discrete random variables with capital letters e.g. X, Y, and their realizations with lower case letters x, y. A random variable X takes values in a set X. We use ||X|| to denote the cardinality of a finite discrete set X, and p_X(x) denotes the probability density function (p.d.f.) of X on X. For brevity we may omit the subscript X when it is obvious from the context. We denote vectors with boldface letters, e.g. x, y. We use T^n(X) to denote the set of ϵ-strongly typical sequences of length n, w.r.t. density p_X(x) on X. Further, we use T^n_o(Y|x) to denote the set of all n-sequence y such that (x, y) are typical, w.r.t. p_XY(x, y). For block coding, we denote the vector x transmits in block j by x[j]. For i < j let T^i_j = {i, i + 1, · · ·, j}. Also for a set S, let X_S = {X_i : i ∈ S} and R_S = ∑_{i∈S} R_i.

B. Relay Network Model

The N-relay network consists of a source with channel input X, N relays where for i-th relay X_i denotes the channel input and Y_i denotes the channel output, and a destination with channel output Y. We denote this network by (X × X_1 × · · · × X_N, p(y, y_1, · · · , y_N|x, x_1, · · · , x_N), Y × Y_1 × · · · × Y_N). We use the same definition for relay network, relay network functions and average probability of error as defined in [5].

III. ACHIEVABLE RATE BASED ON OFFSET DECODING

Let C = {L_1, L_2, · · · , L_K} be an ordered partition on the T = T_1^N such that T = ∪_{i=1}^K L_i. Now we state the main result of this section.
Theorem 1: For a given $C = \{L_1, \cdots, L_K\}$, relay network achieves any rate up to
\begin{equation}
R_{CF} = \sup_p I(X; Y|\hat{Y}|T)
\end{equation}
if for each $S \subseteq T$ the following constraint holds:
\begin{equation}
\sum_{i \in S} I(\hat{Y}_i; Y_i|X_i) \leq \sum_{i \in S} H(X_i|\hat{Y}_i) - \sum_{i=1}^{K+1} H(\hat{Y}_{S_i-1}, X_{L_i}|\hat{Y}_{Li-1}, X_{L_i}^c, \hat{Y}_{Li-1}^c, Y_i)
\end{equation}
where $L_{K+1} = L_0 = \emptyset$, $L_i = \cup_{j=0}^{i-1} L_j$, $S_i = S \cap L_i$, $S_i^c$ is the complement of $S_i$ in $L_i$, and supremum is taken over all joint distribution of the form:
\begin{equation}
p(x, x_T, y_T, \hat{y}_T, y) = p(x) \prod_{i \in T} p(x_i)p(\hat{y}_i|x_i, y_i)p(y_T|y, x_T)
\end{equation}
Before proving Theorem 1, we state two corollaries.

Corollary 1: Compress and Forward achieves any rate up to $R_{CF}$ as (1) if the following constraint holds:
\begin{equation}
I(\hat{Y}_S; Y_S|X_T Y_S Y_C) \leq I(X_S; Y|X_{SC})
\end{equation}
where $\mathcal{S}_C$ is the complement of $S$ in $T$ and for the same joint distribution as (3).

Proof: Put $C = [T]$ in (2). We have:
\begin{equation}
\sum_{i \in S} I(\hat{Y}_i; Y_i|X_i) \leq \sum_{i \in S} H(X_i|Y_i, Y_S Y_C) - H(\hat{Y}_S|X_T Y_S Y_C)
\end{equation}
\begin{equation}
\Leftrightarrow H(\hat{Y}_S|Y_S Y_C, X_T Y_S Y_C) - \sum_{i \in S} H(Y_i|X_i) \leq \sum_{i \in S} H(X_i|Y_S Y_C) - H(\hat{Y}_S|X_T Y_S Y_C)
\end{equation}
\begin{equation}
\Leftrightarrow I(\hat{Y}_S; Y_S|X_T Y_S Y_C) \leq I(X_S; Y|X_{SC})
\end{equation}
where (a) follows from the chain rule for entropy, the LHS of (b) follows from the fact given $(X_i, Y_i)$'s, $Y_i$'s are independent, and the RHS of it follows because $X_i$'s are independent and (c) follows from the fact that given $(X_S Y_C)$, $\hat{Y}_S$ is independent from other random variables and also the fact that $X_S$ and $X_{SC}$ are independent.

Now we compare our result with [2, Theorem 3]. Put in [2, Theorem 3] $U_i = \emptyset$. In other words, we do not have any partial decoding at relay, we obtain that $R_{CF}$ is achievable if the following constraint holds:
\begin{equation}
I(\hat{Y}_S; Y_S|X_T Y_S Y_C) + \sum_{i \in S} I(\hat{Y}_i; X_T Y_{\{i\}}|X_i) \leq I(X_S; Y|X_{SC})
\end{equation}
Now we see that constraint (4) subsumes (6). Hence, we improved the feasibility constraint of [2, Theorem 3] in the case of no partial decoding.

We now show an important property of feasibility region (2) that will be used in the next section.

Corollary 2: The corresponding constraint to $S = T$ is independent of choosing any ordered partition and equivalent with the following constraint:
\begin{equation}
I(\hat{Y}_T; Y_T|X_T) \leq I(X_T; Y)
\end{equation}
Proof: For $S = T$, we have $S_i = L_i$, substituting it in (2)
gives:
\begin{equation}
\sum_{i \in T} I(\hat{Y}_i; Y_i|X_i) \leq \sum_{i \in T} H(X_i|\hat{Y}_i) - \sum_{i=1}^{K+1} H(\hat{Y}_{L_i-1}, X_{L_i}|\hat{Y}_{L_i-1}, X_{L_i}^c, Y_i)
\end{equation}
\begin{equation}
= \sum_{i \in T} H(X_i|\hat{Y}_i) - H(\hat{Y}_T X_T Y_T)
\end{equation}
where (8) follows from chain rule for entropy. Now (8) is independent of $C$ that proves the first part of the corollary. The equivalence of (7) and (8) follows from similar equations used in (5).

Remark 1: Constraint (7) can be regarded as a constraint for single relay channel if we assume that we have an equivalent relay, consists of all relays.

Now we state proof of Theorem 1.

Proof:

Overview of coding strategy: We transmit $B$ messages in $B + K$ blocks of transmission. Code construction at relays are the same as [2, Theorem 3]. Relay $i$ using $x_i$ compresses $y_i$ to $\hat{y}_i$. The destination first decodes $x_{L_i}$, then considers previous block and decodes $(\hat{Y}_{L_i}, x_{L_i})$ simultaneously and this procedure is repeated until destination decodes $\hat{Y}_{L_i}$. Finally destination decodes the source message. In error analysis, we use the fact that codebook generations at relays are independent that enables us to improve the result of [2].

Codebook Generation:

- At transmitter: Generate $2^{nR_t}$ i.i.d. $x(w)$, each drawn uniformly and independently from $T^n_S(X)$, where $w \in [1, 2^nR_t]$.
- At relay $t$: Generate $2^{nR_i}$ i.i.d. $x_t(s_t)$, each drawn uniformly and independently from $T^n_S(X_t)$, where $s_t \in [1, 2^nR_i]$.
- At relay $t$: For each $s_t$, generate $2^{nR_i}$ i.i.d. $\hat{y}_t(z_t, s_t')$, each drawn uniformly and independently from the conditional typical set $T^n_S(Y|s_t(z_t))$, where $z_t \in [1, 2^n(R_t - R_i)]$ and $s_t' \in [1, 2^nR_i]$. This index assignment defines a random and uniform partitioning of a set of size $2^{nR_i}$ into $2^{nR_t}$ subsets with equal size $2^{n(R_t - R_i)}$ that is dubbed parity function in [9].

Encoding: At block $b$

Transmitter in block $1 \leq b \leq B$ sends $x(w_{[b]})$ and for $B < b \leq B + K$ sends $x(1)$. Relay $t$ knows $s_t[b-1]$ from decoding at the end of block $b - 1$ (described below), puts $s_t[b] = s_t'[b-1]$ and sends $x_t(s_t[b])$.

Decoding: At the end of block $b$

1) At relay $t$: The relay knows $(x_t(s_t[b], y_t[b]), s_t[b])$, looks for $(z_t, s_t')$ such that $(\hat{y}_t(z_t, s_t'[b]), x_t(s_t[b], y_t[b]))$ is typical. For large $n$, there exist such $(z_t, s_t')$ denoted by $(z^{[b]}, s^{[b+1]}) = (z_t[b], s_t'[b+1])$ with high probability close to one if
\begin{equation}
\hat{R}_t > I(\hat{Y}_t; Y_t|x_t)
\end{equation}

2) At destination: Decoding at the destination performs in $K + 1$ steps. In step $1 \leq i \leq K + 1$, destination considers block $b - i + 1$ and looks for unique $(\hat{s}_{L_i}, \hat{z}_{L_i})$ such that
\begin{equation}
(\hat{s}_{L_i}, \hat{z}_{L_i}), \hat{Y}_{L_i-1}[b-i+2|\hat{z}_{L_i}[b-i+1]], \hat{Y}_{L_i-1}[b-i+1]|Y[b-i+1]) \in T^c_{L_i}
\end{equation}
where $s_{L_i}[b-i+2]$ is obtained from step $i - 1$ and $(\hat{s}_{L_i}[b-i+2], \hat{Y}_{L_i-1}[b-i+1])$ and $\hat{s}_{L_i}[b-i+1]$ are decoded from previous blocks of decoding. Now, we compute the error probability of this event; i.e there is a $(\hat{s}_{L_i}, \hat{z}_{L_i-1}) \neq (s_{L_i}, \hat{z}_{L_i})$) that satisfies (10). Let $E_i$ be the set of all incorrect pair $(s_{L_i}, \hat{z}_{L_i-1})$. Partition this set into $2|L_i| + |\hat{L}_i| + 1$ subsets
\begin{equation}
E_i = \{ (s_{L_i}, \hat{z}_{L_i-1}) \} \in E_i \cup \{ (s_{L_i}, z_{L_i'}) \} \in \hat{E}_i \cup \{ (s_{L_i}, z_{L_i'}) \} \in \hat{E}_i
\end{equation}
where $V \times U$ is a non-empty subset of $\hat{L}_i \times L_i$ and $\hat{V}, \hat{U}$ are complement of $V, U$ in $\hat{L}_i, L_i$, respectively and $\hat{E}_i, \hat{E}_i$ is the
Now we find the probability that $\mathcal{E}_{\mathcal{U}, \mathcal{V}}$ is non-empty. We have:

$$P(\mathcal{E}_{\mathcal{U}, \mathcal{V}} \neq \emptyset) = \sum_{(s_t, z_v) \in \mathcal{N}_{\mathcal{U}, \mathcal{V}}} P\left[ \left( s_t(s_t), z_v(z_v) : (v \in U, t \in V) \right) \right] \quad (11)$$

where $y_T[b-i+1] = (y, x_{\mathcal{L}^{-1}}, y_{\mathcal{L}^{-1}}[b-i+1])$. Now note that the codebook generation at relays are independent. Hence, each $(\mathcal{S}_t(s_t), \mathcal{S}_v(z_v))$ is drawn uniformly from the set $\mathcal{S}_t = \{ X_{\mathcal{L}[1]}, \ldots, X_{\mathcal{L}[b-K]} \} \times \{ Y_{\mathcal{L}[1]}, \ldots, Y_{\mathcal{L}[b-K]} \}$, where $U = \{ u_1, \ldots, u_p \}$ and $V = \{ v_1, \ldots, v_q \}$. Thus the probability inside (11) is given by:

$$P_r(s_t, z_v) = \frac{1}{2^n} \sum_{u \in U} \sum_{v \in V} \left[ \prod_{i=1}^{K} H(x_i | x_{[b-i]}) \prod_{i=1}^{K} H(y_i | x_{[b-i]}) \right] = 2^{-n} \left( \frac{1}{2} \right)^{2K}$$

The RHS of (19) equals to the following region:

$$\mathcal{R}_S \triangleq \left\{ \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_K \right\} \in \mathbb{R}^{K+1}$$

(18) Now we intend to simplify the region appeared in the RHS of (19). We prove the following theorem:

**Theorem 2:** The RHS of (19) equals to the following region:

$$\mathcal{R}_{CF} = \left\{ (\hat{R}_1, \hat{R}_2, \ldots, \hat{R}_N) : \forall S \subset T \right\}$$

(20) where $S^C$ is the complement of $S$ in $T$. We call this region “CF polytope.”

**Remark 2:** The region $\mathcal{R}_{CF}$ can be obtained from analysis of regular encoding at relays and backward decoding at destination. Thus, we want to obtain backward decoding region with an irregular encoding at relays and offset decoding with less delay respect to backward decoding. This is special case of the general problem of replacing backward decoding with a decoding scheme with less delay [3].

**Proof:** We prove this theorem by strong induction on $N$. For $N = 1$ as base of induction, it is clear. For $N \geq 2$ assume that theorem holds for $n \leq N - 1$. First, it is easy to check that for each $C \in \mathcal{F}_T$ we have $\mathcal{R}_C \supseteq \mathcal{R}_{CF}$, thus first condition of Lemma 1 is satisfied.

Regions $\mathcal{R}_C$ and $\mathcal{R}_{CF}$ are $N - 1$ dimensional polytopes lie in the hyperplane $S_{[b-K]} \cdot \hat{R}_i = \sum_{i \in S} H(x_i | y_i) - H(y_T | X_T)$ with $2N - 2$ facets of dimension $N - 2$ corresponding to maximum value of each $\hat{R}_S$ where $S \subset T, S \neq \emptyset$. Now, we state two lemmas to show...
that polytopes corresponding to each \( C \in \mathcal{F}_T \) satisfy the two next conditions of Lemma 1.

**Lemma 2:** Each facet of \( \mathbf{R}_{CF} \) is covered with the union of some facets of \( \{ \mathbf{R}_C : C \in \mathcal{F}_T \} \).

**Proof:** Suppose \( \mathbf{F}_S \) is the facet of \( \mathbf{R}_{CF} \) corresponding to the maximum value of \( R_S \). Without loss of generality, we show this lemma for \( \mathbf{F}_{T_k} \). Substituting \( S = T_k' \) in (20) yields:

\[
\max(R_{T_k}) = \sum_{t \in T_k'} H(X_t,Y_t) - H(X_{T_k}^t,Y_{T_k}^t|Y_{T_k+1}^t,X_{T_k+1}^t,Y) 
\]

Also by substituting \( S = T_k^b \cup U \), where \( U \subseteq T_{T_k+1}^N \) and maximum value of \( R_{T_k^b} \) in (20), we have,

\[
\hat{R}_U \leq \sum_{t \in U \cap T_k^b} H(X_t,Y_t) - H(X_t,Y_t|X_{T_k^b}^t,Y_{T_k^b}^t) 
\]

where \( U^C \) is the complement of \( U \) in \( T_{T_k+1}^N \) and equality holds for \( U = T_{T_k+1}^N \). Thus the projection of \( \mathbf{F}_{T_k} \) on \( R_{T_k+1}^N \) is also a CF polytope for \( T' = T_{T_k+1}^N \) with \( N-k \) elements. Therefore from induction assumption, we have \( \mathbf{R}_{CF} \cup \mathcal{S}^C \). (21) implies that destination can first decode \( X_{T_k+1}^N, Y_{T_k+1}^N \) reliably without help of any other relays, then decodes other relays information using \( Y' = (X_{T_k+1}^N, Y_{T_k+1}^N) \). Without loss of generality, we assume \( Y' \) as the new output at the destination. Now for each \( V \subseteq T_k^b \), we can write (20) in the following form:

\[
\hat{R}_V \leq \sum_{t \in V} H(X_t,Y_t) - H(X_t,Y_t|X_{T_k}^t,Y_{T_k}^t, Y_{T_k+1}^t) 
\]

where \( V^C \) is the complement of \( V \) in \( T_k^b \) and equality holds for \( V = T_k^b \). Thus the projection of \( \mathbf{F}_{T_k} \) on \( R_{T_k}^N \), is also a CF polytope for \( T'' = T_k^b \) with \( k \) elements. Again from induction assumption, we have \( \mathbf{R}_{CF} \cup \mathcal{S}^C \). On the other hand, it is easy to check that if for each subset \( S(\mathcal{U} \cup V), \mathcal{U} \subseteq T_{T_k+1}^N, V \subseteq T_k^b \) of \( T_k \), (21) and (22) hold, then (20) is satisfied. So we can write \( \mathbf{F}_{T_k} \) as the Cartesian product of \( \mathbf{R}_{CF} \) and \( \mathbf{R}_{CF}^C \). It yields:

\[
\mathbf{F}_{T_k} = \bigcup_{C' \in \mathcal{F}_{T''}, C'' \in \mathcal{F}_{T''}} \mathbf{R}_{C'} \times \mathbf{R}_{C''} \tag{23}
\]

On the other hand, we construct an *ordered partition* from each \( C', C'' \) by setting \( C = [C', C''] \). Now similar equation to (21) based on (18) implies that the facet of polytope \( \mathbf{R}_C \) corresponding to \( T_k^b \) is equal to Cartesian product of \( \mathbf{R}_{C'} \) and \( \mathbf{R}_{C''} \). Therefore we can write \( \mathbf{F}_{T_k} \) as the union of some facets of \( \{ \mathbf{R}_C : C \in \mathcal{F}_T \} \).

**Lemma 3:** For every \( C' \in \mathcal{F}_T \) and for each facet of corresponding polytope, there is a \( C'' \in \mathcal{F}_T \) such that the polytope corresponding to \( C_2 \) has only that facet as a common part.

**Proof:** We omit the complete proof, instead, we give the main ideas behind the proof.

The first idea is inspired by the MAC with inputs \( (X_1, X_2) \) and channel output \( Y \). It is well-known that every point inside achievable region is achieved using joint decoding of \( (X_1, X_2) \). However the corner point \( I(X_1;Y), I(X_2;Y|X_1) \) is achievable using sequential decoding instead of joint decoding when \( X_1 \) is decoded prior to \( X_2 \). Next consider the \( N \)-user multiple access channel with inputs \( X \).

Suppose \( R_{MAC} \) is the boundary of achievable rate region such that \( R_{TF} = I(X_T;Y) \) on it. Now substituting \( Y_t = \emptyset \) in (20) gives \( R_{MAC} = R_{CF} \). The proof of Lemma 2 states that the points on the facet corresponding to maximum value of \( R_S \) is achievable using sequential decoding when first, \( X_{SC} \) are decoded simultaneously and then \( Y_S \) are decoded simultaneously using \( X_{SC} \). From this examples it is intuitively, expected that in the joint decoding of some random variables \( (X_{i4}, Y_{i4}) \) using some information \( Z \), the maximum of \( R_{TF} \) is achievable using sequential decoding when \( Y_2 \) is decoded first in order to provide more information to decode \( X_U \).

The second idea is motivated by the following simple example. Consider a 2-relay network. The destination uses two different sequential decoding schemes. In the first scheme, the destination first decodes \( X_1 \) in block \( b \), then decodes \( X_2 \) in block \( b \), \( Y_1 \) in block \( b - 1 \), and finally decodes \( Y_2 \) in block \( b - 2 \). In the second scheme, destination first decodes \( X_1 \) in block \( b \), then decodes \( X_2 \) in block \( b - 1 \), \( Y_1 \) in block \( b - 1 \), and finally decodes \( Y_2 \) in block \( b - 2 \). These two sequential decoding are special cases of offset decoding using \( [1,2] \) and \( [1], [2] \). Now, it is obvious that two schemes give the same result. In the general case, if the order of decoding of two decoding schemes are the same, the block of decoding has no effect on the error analysis.

Consider \( S \) is a subset of \( T \) such that \( S \not= \cup_{i \in \mathcal{L}_i} L_i, 2 \leq j \leq K \). If \( S = \cup_{i \in \mathcal{L}_i} L_i \), substituting \( S \) in (18) implies that the facet corresponding to maximum value of \( R_S \) lies in \( \mathcal{F}_S \) and hence, is not
inside $\mathbf{R}_{CF}$. Set $\mathbf{C}_2 = [L_{21}, \ldots, L_{2K+1}]$, where $L_{2i} = S_i' \cup S_{i-1}$ ($S_{i-1}$ and $S_i'$ are defined in Theorem 1). Suppose $\mathbf{F}_{C,S}, \mathbf{F}_{C_2,S'}$ to be facets that are corresponding to maximum of $R_S, R_{SC}$ on the $\mathbf{R}_C, \mathbf{R}_{C_2}$, respectively. We justify that these facets must be equal.

Note that the $R_S = \sum_{i=1}^{K+1} R_{S_i} + R_{S_{i-1} - R_{S_{i-1}}}$, so in order to maximize $R_S$, we must maximize $R_j = R_{S_i} + R_{S_{i-1}} - R_{S_{i-1}}$. But $R_j$ represents the rate of joint decoding of $(X_{S_i}, Y_{S_{i-1}})$ with $(X_{S_i}', Y_{S_{i-1}'})$ in step $j$ of decoding at destination. Hence, from the above discussion, $R_j$ is maximized when destination decodes $(X_{S_i}', Y_{S_{i-1}'})$ prior to $(X_{S_i}, Y_{S_{i-1}})$ in step $j$ of decoding.

It induces the following order on the decoding at the destination (without noting to the block of decoding).

$$X_{S_1}', X_{S_1}, \ldots, (X_{S_i}', Y_{S_{i-1}'}, Y_{S_{i-1}}), (X_{S_i}, Y_{S_{i-1}}), \ldots, Y_{S_K}'$$

(24)

A similar argument shows that we can achieve the maximum of $R_{SC}$ on $\mathbf{R}_{C_2}$ when we have the following order of decoding at destination:

$$X_{S_1}'', X_{S_1}'', \ldots, (X_{S_j}'', Y_{S_{j-1}'}, Y_{S_{j-1}}), (X_{S_j}, Y_{S_{j-1}}), \ldots, Y_{S_{K+1}}$$

(25)

where $S_j'' = S'' \cap L_{2j} = S_j', S_2'' = S_1'' = S_{j-1}$ and $S_{j+1}'' = S''_{j+1} = \phi$. Now (24) and (25) are the same. Because the order of decoding is the same in both decoding schemes, for each $t \in T$ the bin index $X_t$ is decoded prior to $Y_t$; we conclude that $\mathbf{F}_{C,S} = \mathbf{F}_{C_2,S'}$. Finally $\mathbf{F}_{C_2,S'}$ is corresponding to minimum of $R_S$ on $\mathbf{R}_{C_2}$ (since $R_S + R_{SC}$ is constant). Therefore $\mathbf{R}_{C} \cap \mathbf{R}_{C_2} = \mathbf{F}_{C,S}$. A direct proof can be made from evaluating $\mathbf{F}_{C,S} \mathbf{F}_{C_2,S'}$ by using (18).

Proof of Theorem 2 (continued):

Now Lemma 2 and Lemma 3 showed that $\mathbf{R}_{CF}$ and $\{\mathbf{R}_C : C \in \mathcal{F}_T\}$ satisfy the conditions of Lemma 1, hence we have:

$$\mathbf{R}_{CF} = \bigcup_{C \in \mathcal{F}_T} \mathbf{R}_C$$

(26)

Thus, we prove theorem for $N$ and induction is completed.

Examples: Fig 1 and Fig 2 illustrate Theorem 2 for two and three relay networks, respectively. For two-relay network, there are three ordered partitions, hence $\mathbf{R}_{CF}$ is covered with three segments. Fig 2 only shows the boundary of feasibility region ($\mathbf{R}_{CF}$). As expected from Lemma 2, each edge of this hexagon is covered with three segments (each edge(facet) is corresponding to a maximum or minimum of $R_i$ : $1 \leq i \leq 3$, this reduces the edge constraint to feasibility region of two-relay network). Also, this figure shows that each edge(facet) of $\mathbf{R}_{CF}$ inside $\mathbf{R}_{CF}$ is the common part of $\mathbf{R}_C$, with one $\mathbf{R}_{CF}, j \neq i$. Additionally, this figure shows that some $\mathbf{R}_C$ is not hexagon. Assume $C = C_1$. We show that $\mathbf{F}_{C_1,(2)}$ consists of a one point. It follows from the proof of Lemma 3 that to achieve a point on $\mathbf{F}_{C_1,(2)}$, destination must consider the following order of decoding:

$$X_3, Y_3, X_2, X_1, Y_2, Y_1$$

So, we have one choice for decoding and therefore have only one point on $\mathbf{F}_{C_1,(2)}$. Next consider $\mathbf{F}_{C_1,(3)}$. To achieve a point on this facet, destination must consider the following order of decoding:

$$X_3, X_2, Y_3, \{X_1, Y_2\}, Y_1$$

Here, because destination has the option of joint decoding of $\{X_1, Y_2\}$, $\mathbf{F}_{C_1,(3)}$ is a segment.

Now, we are ready to state the main theorem of the paper as follows:

**Theorem 3:** Compress-and-forward achieves any rate up to

$$R_{CF} = I(X; Y_t|X_P)$$

where

$$I(Y_S; X'_{SY} X_{SC} Y) \leq I(X_S; Y_{SY} S_{SC} X_{SC})$$

(27)

Now for all $S \subseteq T$ and any joint distribution of the form (3).

**Proof:** This theorem is straightforward result of (19) and Theorem 2. We have:

$$\sum_{i \in S} I(Y_i; Y_{i}|X_i) \leq \sum_{i \in S} H(X_i Y_i) - H(X_S Y_S Y_{SC} X_{SC})$$

(28)

where (a) follows from definition of $R_{CF}$, (b) follows from similar equations used in (5).

C. Source-Channel Coding Separation Interpretation

Constraint (28) has a simple source-channel coding separation interpretation. Let $\Lambda = (S; S''_T, D)$ be a cutset(pair) of all relays and destination. Now the RHS of (28) represents the maximum flow through the boundary of $\Lambda$ by knowing $(Y_{SC}, X_{SC})$ at destination. The LHS of (28) represents the maximum rate of compression of $Y_S$ to $Y_S$ by knowing $(Y_{SC}, X_{SC}, X_S)$ at destination (Note that $X_S$ is decoded prior to $Y_S$ as bin indexes). Now the rate of compression must be below of the maximum rate of transmission in order to transmit $Y_T$ as sources to destination reliably, in the view of source channel separation.

V. CONCLUSIONS

In this paper, we generalized the well-known compress and forward strategy for relay networks. For encoding, this strategy uses irregular encoding at all relays. For decoding, destination considers an ordered partition on the set of relays and implements an offset decoding. In error analysis, we use the fact that code-book generation at relays are independent. It enables us to achieve CF rate with more flexible constraints. A covering lemma is utilized to unify the feasibility regions of all ordered partition in the closed form. It is shown that the unified feasibility region satisfies a source-channel coding separation constraint.

REFERENCES


